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The energy balance in modulated plane Poiseuille flow

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Plane Poiseuille flow in which the pressure gradient has a small amplitude time-periodic component in addition to a constant component is considered. The velocity field close to the boundaries, arising from a small amplitude high frequency disturbance to the flow, is calculated to second order in the modulation amplitude. The energy-transfer integral for the disturbance is then calculated to the same order. It is found that, if the thickness of the disturbance shear wave relative to that of the modulation shear wave is greater than $\frac{1}{2}$, the modulation inhibits energy transfer into the disturbance and so stabilizes the flow.

1. Introduction

The experimental study by Donnelly (1964) of the stability of modulated Couette flow stimulated several attempts to reproduce his results theoretically (Venezian 1969; Rosenblat & Herbert 1970; Rosenblat & Tanaka 1971). There have also been theoretical studies of the effects of modulation on the linear stability of other flows (see, for example, Grosch & Salwen 1968).

The results of these analyses reveal that, in the problems considered, modulation has a stabilizing effect. The degree of this stabilization initially increases and then decreases as the modulation frequency is increased from zero. Thus, an optimum frequency exists at which the enhancement of stability by modulation is a maximum. This feature is also present in Donnelly's results.

An attempt to gain some understanding of the mechanism of this stabilization appears to have been made only by Grosch & Salwen (1968) in their study of modulated plane Poiseuille flow. In this work the pressure gradient has constant and time-periodic components. Grosch & Salwen found that the flow is most stabilized when the shear waves associated with the modulation and the disturbance are of comparable thickness. They suggested that these shear waves interact to inhibit the transfer of energy from the base flow to the disturbance and so stabilize the flow. To verify this, they attempted to extend Lin's (1954) work on unmodulated parallel flows to cover modulated plane Poiseuille flow.

Lin considered high-frequency disturbances to parallel flows and determined the distribution of Reynolds stress within the resulting wall Stokes layers. He found that, in these flow regions, the Reynolds stress τ_0 is such as to transfer energy from the base flow into the disturbance and so destabilize the flow. Grosch & Salwen calculated the Reynolds stress in the wall Stokes layers of the modulated flow in the form $\tau = \tau_0 + \epsilon^2\tau_2 + O(\epsilon^3)$, where ϵ is the amplitude of the modulation. No details of these calculations are given.

From the energy equation governing a disturbance to a time-dependent flow, it is evident that energy transfer in modulated parallel flow arises from an interaction between the time-dependent base flow and the disturbance. It follows that a calculation of the Reynolds stress as given by Grosch & Salwen, in which the time dependence of the base flow is ignored, is not appropriate in a discussion of the energy balance in modulated parallel flows.

Under the same approximations as those used by Lin, the energy transfer integral is calculated to second order in ϵ. It is found that a sufficient condition for the modulation to be stabilizing is that $\delta/\delta_m \geqslant \frac{1}{2}$, where δ_m and δ are the thicknesses of the modulation and disturbance shear waves.

2. The base flow

The two-dimensional flow of an incompressible viscous fluid between two infinite horizontal boundaries is considered. Cartesian co-ordinates (x, y) are used with origin in the lower boundary and $x > 0$ in the flow direction.

The flow is driven by a pressure gradient

$$\frac{\partial P}{\partial x} = -\frac{8\rho\overline{U}\nu}{l^2}(1-\epsilon'\cos\omega_m t),$$

where ρ and ν are the density and kinematic viscosity of the fluid, $\overline{U}$ is the maximum velocity of the unmodulated flow, l is the distance between the horizontal boundaries, ϵ' is the non-dimensional amplitude of the modulation, ω_m is the frequency of the modulation and t is the time.

The equation for the velocity $U(y, t)$ of the base flow is

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 U}{\partial y^2}$$

and the boundary conditions are $U(0, t) = U(l, t) = 0$. The appropriate high-frequency solution of this equation is

$$U(y,t) = 4\overline{U}\left[\eta(1-\eta)+\frac{\epsilon'}{\Omega_m^2}(\sin\omega_m t+\mathrm{Re}\{i\exp[-(1+i)\,\Omega_m\eta+i\omega_m t]\})\right], \quad (1)$$

where $\eta = y/l$ and $\Omega_m = (\omega_m l^2/2\nu)^{\frac{1}{2}} \gg 1$.

The modulation of the pressure gradient is seen to produce a uniform velocity component exactly out of phase with the modulation together with a shear wave which is confined to the Stokes layer adjacent to the boundaries of thickness $\delta_m = (2\nu/\omega_m)^{\frac{1}{2}}$.

Since $\Omega_m \gg 1$, a large modulation pressure amplitude ϵ' produces only a small effect on the base flow provided that $\epsilon' \ll \Omega_m^2$. However, if $\epsilon' = 1$, an inflexion point in the base velocity profile appears at $y = 0$. As we are interested here in Tollmien–Schlichting instabilities, we take $\epsilon' < 1$ and so $\epsilon = \epsilon'/\Omega_m^2 \ll 1$.

3. The disturbed flow

Following Lin (1954), we consider a small amplitude disturbance to the flow in the form of a high-frequency wave. Such a disturbance generates vorticity

$$\zeta(x,y,t) = (\partial v/\partial x) - (\partial u/\partial y), \tag{2}$$

which satisfies the linear equation

$$\frac{\partial \zeta}{\partial t} - v\frac{\partial^2 U}{\partial y^2} + U\frac{\partial \zeta}{\partial x} = \nu\nabla^2\zeta, \tag{3}$$

where u and v are the disturbance velocity components and

$$\nabla^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2).$$

Substitution of the expansions

$$\left.\begin{aligned} \zeta(x,y,t) &= Z_0(x,y,t) + \epsilon Z_1(x,y,t) + \epsilon^2 Z_2(x,y,t) + O(\epsilon^3), \\ u(x,y,t) &= u_0(x,y,t) + \epsilon u_1(x,y,t) + O(\epsilon^2), \\ v(x,y,t) &= v_0(x,y,t) + \epsilon v_1(x,y,t) + O(\epsilon^2), \\ U(y,t) &= U_0(y) + \epsilon U_1(y,t) \end{aligned}\right\} \tag{4}$$

into (3) leads to the system of equations

$$\frac{\partial Z_0}{\partial t} - v_0\frac{d^2U_0}{dy^2} + U_0\frac{\partial Z_0}{\partial x} = \nu\nabla^2 Z_0, \tag{5}$$

$$\frac{\partial Z_1}{\partial t} - v_0\frac{\partial^2 U_1}{\partial y^2} - v_1\frac{d^2U_0}{dy^2} + U_0\frac{\partial Z_1}{\partial x} + U_1\frac{\partial Z_0}{\partial x} = \nu\nabla^2 Z_1, \tag{6}$$

$$\frac{\partial Z_2}{\partial t} - v_1\frac{\partial^2 U_1}{\partial y^2} - v_2\frac{d^2U_0}{dy^2} + U_0\frac{\partial Z_2}{\partial x} + U_1\frac{\partial Z_1}{\partial x} = \nu\nabla^2 Z_2, \ldots. \tag{7}$$

$O(1)$ velocity components

In the viscous sublayers immediately adjacent to the boundaries the dominant process is one of diffusion of vorticity, so that (5) reduces to

$$\mathscr{L}Z_0 = 0, \tag{8}$$

where $\mathscr{L} \equiv \partial/\partial t - \nu\, \partial^2/\partial y^2$. This reduction is valid provided that

$$\overline{U}/c\Omega^2 \ll 1, \quad \overline{U}/c\Omega_m \ll 1, \quad \alpha\delta \ll 1, \tag{9a}$$

where α is the wavenumber, $\delta = (2\nu/\omega)^{\frac{1}{2}}$ the thickness, ω the frequency and $c = \omega/\alpha$ the speed of the disturbance shear wave and $\Omega = l/\delta = (\omega l^2/2\nu)^{\frac{1}{2}}$. [It should be noted that, within the wall Stokes layer, $y = O(\delta_m)$ and so $U_0 = O(\overline{U}\delta_m/l)$ and not $O(\overline{U})$.]

We assume now that the modulation and disturbance shear waves are of comparable thickness so that $\Omega = O(\Omega_m)$. As $\Omega_m \gg 1$ the conditions (9a) simplify to

$$\overline{U}/c\Omega_m \ll 1, \quad \alpha\delta \ll 1. \tag{9b}$$

For the least stable mode in unmodulated plane Poiseuille flow, Ω is approximately 10^2 in the neighbourhood of the minimum Reynolds number on the neutral stability curve. Hence the assumption that $\Omega \gg 1$ for Tollmien–Schlichting instabilities is well justified.

Equation (8) has the solution

$$Z_0(x,y,t) = \mathrm{Re}\,[A\, e^{i(\alpha x-\omega t)}\, e^{-(1-i)\xi}], \tag{10}$$

where $\xi = y/\delta = \Omega\eta$ and A is a complex constant with the dimensions of vorticity.

The $O(1)$ disturbance velocity components are found from the appropriate forms of the continuity equation and equation (2):

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y} = 0, \quad \frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y} = Z_0(x,y,t).$$

The solution of these equations which has $u_0(x,\,0,t) = v_0(x,0,t) = 0$ is

$$u_0(x,y,t) = \mathrm{Re}\,[A\delta e^{i(\alpha x-\omega t)} f(\xi)], \quad v_0(x,y,t) = \mathrm{Re}\,[A\alpha\delta^2 e^{i(\alpha x-\omega t)} g(\xi)], \tag{11}$$

where $$f(\xi) = \tfrac{1}{2}(1+i)\,[-1+e^{-(1-i)\,\xi}]$$

and $$g(\xi) = \tfrac{1}{2}[1-(1-i)\,\xi-e^{-(1-i)\,\xi}].$$

$O(\epsilon)$ velocity components

Under conditions (9), equation (6) reduces to

$$\mathscr{L}Z_1 = v_0\frac{\partial^2 U_1}{\partial y^2}-U_1\frac{\partial Z_0}{\partial x}. \tag{12}$$

We require not the exact solution of this equation but only the first few terms in the expansion of $Z_1(x,y,t)$ in powers of ξ for $\xi \ll 1$.

After substitution for the terms on the right-hand side of (12), the solution is found to be

$$Z_1(x,y,t) = \mathrm{Re}\,[B\, e^{i(\alpha x-\omega t)}\{e^{i\omega_m t}(\alpha_1+\alpha_2\xi+O(\xi^2))+e^{-i\omega_m t}(\beta_1+\beta_2\xi+O(\xi^2))\}], \tag{13}$$

where $B = A\overline{U}/c$ and

$$\alpha_1 = -\frac{4L^4-L^2-1}{2L}+i\frac{L^4-1}{L^2},$$

$$\alpha_2 = \frac{1}{2L^2}[(4L^6+2L^5-3L^4-L^3-L^2-L+2) + i(4L^6-2L^5-3L^4+L^3-L^2+L+2)],$$

$$\beta_1 = \frac{i}{2L^2}(4L^5+2L^4+L^3-L-2),$$

$$\beta_2 = -\frac{1+i}{2L^2}(4L^6+2L^5+3L^4+L^3-L^2-L-2);$$

$L = \delta/\delta_m = \Omega_m/\Omega = (\omega_m/\omega)^{\frac{1}{2}}$ is taken to be of $O(1)$.

The $O(\epsilon)$ velocity components (u_1, v_1) satisfy the equations

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y} = 0, \quad \frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y} = Z_1(x,y,t).$$

The required solution of these equations is

$$\left.\begin{aligned} u_1(x,y,t) &= \mathrm{Re}\,[-B\delta\, e^{i(\alpha x-\omega t)}\{e^{i\omega_m t}(\alpha_1\xi+\tfrac{1}{2}\alpha_2\xi^2+O(\xi^3)) \\ &\qquad + e^{-i\omega_m t}(\beta_1\xi+\tfrac{1}{2}\beta_2\xi^2+O(\xi^3))\}], \\ v_1(x,y,t) &= \mathrm{Re}\,[B\alpha\delta^2\, e^{i(\alpha x-\omega t)}\{e^{i\omega_m t}\,(\tfrac{1}{2}i\alpha_1\xi^2+\tfrac{1}{6}i\alpha_2\xi^3+O(\xi^4)) \\ &\qquad + e^{-i\omega_m t}\,(\tfrac{1}{2}i\beta_1\xi^2+\tfrac{1}{6}i\beta_2\xi^3+O(\xi^4))\}]. \end{aligned}\right\} \tag{14}$$

$O(\epsilon^2)$ velocity components

The simplified form of (7) is

$$\mathscr{L}Z_2 = v_1\frac{\partial^2 U_1}{\partial y^2}-U_1\frac{\partial Z_1}{\partial x}. \tag{15}$$

Hence $Z_2 = O(B\overline{U}/c)$. We shall show that as a consequence of this the solution of (15) is not required.

4. The energy equation

The energy equation for modulated plane Poiseuille flow is

$$\frac{\partial}{\partial t}\iint\langle\tfrac{1}{2}(u^2+v^2)\rangle\,dx\,dy = \iint\left\langle -uv\frac{\partial U}{\partial y}\right\rangle dx\,dy-\nu\iint\langle\zeta^2\rangle\,dx\,dy, \tag{16}$$

or

$$\partial E/\partial t = I_1-\nu I_2,$$

where the integration with respect to x is an average over a wavelength $2\pi/\alpha$ of the disturbance, that with respect to y is from $y = 0$ to $y = l$, and

$$\langle f(x,y,t)\rangle = \frac{\omega_m}{2\pi}\int_0^{2\pi/\omega_m} f(x,y,t)\,dt.$$

Equation (16) gives the time rate of change of the disturbance energy E in terms of a balance between the energy transfer integral I_1 and the viscous dissipation integral I_2.

We take now

$$u(x,y,t) = \mathrm{Re}\,[e^{i(\alpha x-\omega t)}\hat{u}(\xi,\omega_m t)],\quad v(x,y,t) = \mathrm{Re}\,[e^{i(\alpha x-\omega t)}\hat{v}(\xi,\omega_m t)] \tag{17}$$

and

$$\frac{\partial U}{\partial y} = \frac{dU_0}{dy}+\epsilon\left[\frac{\partial U_{11}}{\partial y}e^{i\omega_m t}+\frac{\partial U_{12}}{\partial y}e^{-i\omega_m t}\right],$$

where

$$\begin{aligned} \hat{u}(\xi,\omega_m t) &= u_{00}(\xi)+\epsilon[u_{11}(\xi)\,e^{i\omega_m t}+u_{12}(\xi)\,e^{-i\omega_m t}] \\ &\qquad +\epsilon^2[u_{20}(\xi)+u_{21}(\xi)\,e^{i\omega_m t}+u_{22}(\xi)\,e^{-i\omega_m t}]+O(\epsilon^3) \end{aligned}$$

and

$$\begin{aligned} \hat{v}(\xi,\omega_m t) &= v_{00}(\xi)+\epsilon[v_{11}(\xi)\,e^{i\omega_m t}+v_{12}(\xi)\,e^{-i\omega_m t}] \\ &\qquad +\epsilon^2[v_{20}(\xi)+v_{21}(\xi)\,e^{i\omega_m t}+v_{22}(\xi)\,e^{-i\omega_m t}]+O(\epsilon^3). \end{aligned}$$

Substitution of (17) in the energy-transfer integral gives

$$I_1 = -\frac{1}{2}\int\left\langle \mathrm{Re}\left[\hat{u}\hat{v}^*\frac{\partial U}{\partial y}\right]\right\rangle dy, \tag{18}$$

where * denotes a complex conjugate. Further,

$$\left\langle \hat{u}\hat{v}^* \frac{\partial U}{\partial y} \right\rangle = u_{00} v_{00}^* \frac{dU_0}{dy} + \epsilon^2 \left[(u_{00} v_{12}^* + u_{11} v_{00}^*) \frac{\partial U_{12}}{\partial y} + (u_{00} v_{11}^* + u_{12} v_{00}^*) \frac{\partial U_{11}}{\partial y} + (u_{00} v_{20}^* + u_{20} v_{00}^* + u_{11} v_{11}^* + u_{12} v_{12}^*) \frac{\partial U_0}{\partial y} \right] + O(\epsilon^3). \quad (19)$$

Grosch & Salwen appear to have neglected all but the last two of the $O(\epsilon^2)$ terms in (19) and so have completely ignored the contribution to the energy-transfer integral arising from the interaction of the disturbance and the unsteady part of the base flow. Further, of the $O(\epsilon^2)$ terms in (19), the relative magnitude of the first four terms with respect to the remaining terms is $c\Omega_m/\overline{U}$. It follows from (9) that, to $O(\epsilon^2)$, the energy transfer arising from the interaction of the disturbance and the steady part of the base flow is negligibly small compared with that arising from the interaction involving the unsteady part of the base flow.

Also, we see now, that, to evaluate the energy-transfer integral to $O(\epsilon^2)$, we require the disturbance vorticity to $O(\epsilon)$ only and so (15) may be discarded.

From (1) and (11) we have

$$\frac{\partial U_{11}}{\partial y} = \frac{2\overline{U}}{l\Omega_m}[-(1-i) + 2L\xi + O(\xi^2)],$$

$$\frac{\partial U_{12}}{\partial y} = \frac{2\overline{U}}{l\Omega_m}[-(1+i) + 2L\xi + O(\xi^2)],$$

$$u_{00}(\xi) = A\delta[-\xi + \tfrac{1}{2}(1-i)\xi^2 + O(\xi^3)],$$

and

$$v_{00}(\xi) = A\alpha\delta^2[\tfrac{1}{2}i\xi^2 - \tfrac{1}{6}(1+i)\xi^3 + O(\xi^4)].$$

Substitution of these expansions, together with those for the first-order velocity components given in (14), into (18) gives

$$I_1 = 4\frac{\overline{U}}{l}\int \tau_0 \left[1 - \epsilon^2 \left\{ \frac{\overline{U}}{c\Omega_m}(L+1)(2L-1)(2L^2-L+1) + O\left(\left(\frac{\overline{U}}{c\Omega_m}\right)^2\right) \right\} + O(\epsilon^3)\right] dy, \quad (20)$$

where

$$\tau_0 = \frac{\nu A^2}{3}\frac{\overline{U}}{c}\frac{\xi^4}{\Omega} + O(\xi^5)$$

is the Reynolds stress as found by Lin. Finally, the dissipation integral

$$I_2 = \iint \langle \zeta^2 \rangle \, dx\, dy = \mathrm{Re}\{\iint (Z_0^2 + \epsilon^2 \langle Z_1^2 \rangle + O(\epsilon^3))\, dx\, dy\}. \quad (21)$$

Hence, if $L > \frac{1}{2}$ the energy transfer from the base flow to the disturbance in the wall Stokes layer is decreased. As the viscous dissipation of the disturbance energy is increased by the presence of modulation, it follows that, within the wall Stokes layer, modulation makes the flow more stable. Further, if the principal contribution to both energy-transfer and dissipation integrals occurs within the wall Stokes layer then the condition $L \geqslant \frac{1}{2}$ is a sufficient condition for the entire flow to be made more stable.

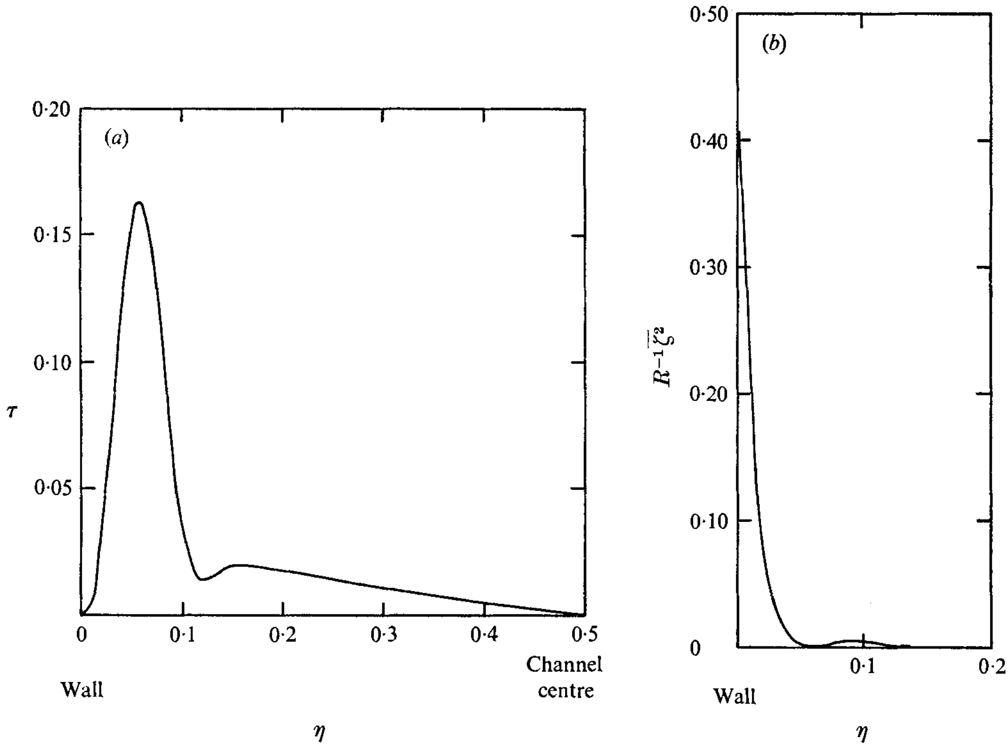

FIGURE 1. Distribution of (*a*) Reynolds stress and (*b*) dissipation in unmodulated plane Poiseuille flow at $R = 2 \times 10^4$.

From the numerical results of Thomas (1953) for unmodulated plane Poiseuille flow, the distribution of Reynolds stress and dissipation can be found for the particular case $R = \overline{U}l/\nu = 2 \times 10^4$, $\alpha l = 2$ and $c/\overline{U} = 0{\cdot}2375 - 0{\cdot}00359i$. These distributions are shown in figures 1 (*a*) and (*b*). The critical point at which $\text{Re}\,[c/U] = 1$ is at $y = 0{\cdot}06l$. Figure 1 (*b*) reveals that the dissipation occurs almost totally within the wall Stokes layer while, from figure 1 (*a*), it may be seen that the energy transfer has a peak very close to the critical point. As the mean position of the critical point is unchanged by the presence of modulation, we can assert that the condition $L \geqslant \frac{1}{2}$ is sufficient for modulated plane Poiseuille flow to be more stable than the corresponding unmodulated flow.

While it is not possible to determine the value of L for which this stabilization is greatest, this result is not inconsistent with the suggestion by Grosch & Salwen that the maximum stabilization of modulated plane Poiseuille flow occurs when the disturbance and modulation shear waves are of comparable thickness.

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